THERMODYNAMIC NOISE LIMITS OF THE SIMPLE SPRING-MASS GRAVIMETER

M. E. Gruchalla, June 8, 1994, August 10, 1995

One of the fundamental first-principle mechanisms limiting measurements in any physical system is the thermodynamic noise of the system. There are also a number of non-statistical noise mechanisms that introduce excess noise, that is noise in excess of the first-principle noise, that will further limit measurement sensitivity. This work considers only the limitations imposed by thermodynamics and specifically does not consider any excess-noise limitations. In all physical systems at a temperature above absolute zero, the temperature of the system introduces a thermal noise power into the system. This thermal noise power is a fundamental first-principle mechanism and cannot be eliminated or otherwise avoided. The thermal noise power introduces an uncertainty into all observations in the system, and therefore places a lower limit on the precision of any measurement. One traditional gravimeter configuration is shown in Figure 1 which comprises a proof mass m suspended by a suitable spring of spring constant k_s with damping element b to damp periodic excitations adequately to allow precision observation. The observed parameter is the position of the proof mass as a function of the gravimetric force exerted on the mass causing a displacement against the spring constant of the spring. Thermal noise power causes the proof mass to move with a random motion. This thermal-noise displacement sets a lower bound on the precision to which the position of the proof mass many be observed. This thermodynamic noise-limited sensitivity of a simple damped spring-mass gravimeter is derived below based on first-principles. Such excess noise as non-statistical variations and creep in the spring, and systematic noise such as respiration of the proof mass (absorption and release of gas causing changes in the actual total mass of the proof mass), buoyancy, tidal effects, cultural noise, etc. specifically are not considered.

The thermal-noise displacement may be very easily computed from equipartition theorem of statistical mechanics[1] and specifically Brownian movement.[2] According to Boltzmann, the kinetic energy of a particle due to thermal excitation is $kT/2$ for each degree of freedom where k is Boltzmann's constant (this must not be confused with the spring constant k_s) and T is the absolute temperature in °K. This holds for individual particles such as atoms as well as for large particles such as the mass of the spring-mass system which is a collection of smaller particles. In the case of a collection of particles, the calculations are referred to the center of mass of the collection. Thermal noise is a random process, so peak values of noise parameters have no meaning. The values of noise parameters must be expressed in terms of their root-mean-square (rms) values. If v_{rms} is defined as the rms velocity of the mass in the simple spring-mass system, the average kinetic energy of the mass is simply $mv_{rms}^2/2$. Similarly, if x_{rms} is the rms displacement of the mass, and the spring as well, the average potential energy is $k_s x_{rms}^2/2$. Since the system is a simple harmonic oscillator, the average kinetic energy must equal the average potential energy. Therefore, the average potential energy must equal the thermal-excitation energy.

$$\textit{Thermal Kinetic Energy} = \frac{kT}{2}$$

where: k = Boltzmann's Constant

T = Absolute Temperature in °K

$$\textit{Average Mass Kinetic Energy} = \frac{mv_{rms}^2}{2}$$

where: m = System Dynamical Mass (Effective Proof Mass)

v_{rms}^2 = Mean-Square Mass Velocity

$$\textit{Average Spring Potential Energy} = \frac{k_s x_{rms}^2}{2}$$

where: k_s = Spring Constant

x_{rms}^2 = Mean-Square Mass Displacement

For convenience, define the parameter ω_0 which will later be shown to be the undamped natural frequency.

$$\omega_0^2 \equiv \frac{k_s}{m} \tag{1}$$

Then, from Boltzmann;

$$\frac{kT}{2} = \frac{mv_{rms}^2}{2} = \frac{k_s x_{rms}^2}{2}$$

$$x_{rms}^2 = \frac{kT}{k_s} \tag{2a}$$

Solving Equation (1) for k_s and substituting into Equation (2a),

$$x_{rms}^2 = \frac{kT}{m\omega_0^2} \tag{2b}$$

This result of Equation (2a) is counter-intuitive. The thermal noise displacement limit is a function only of the system temperature and the spring constant — rms thermal-noise displacement is directly related to the square root of the absolute temperature of the system and inversely related to the square root of the spring constant. The mass and the damping coefficient do not affect the limiting thermal-noise displacement. For example, a k_s may be chosen for some desired thermal-noise displacement at some specific temperature. Then, any desired natural frequency and damping may be obtained without affecting the system noise by computing the needed mass and damping coefficient. Therefore, the system thermal-noise displacement may be lowered only by lowering the system temperature or by increasing the spring constant.

The results of Equation (2) were developed directly from the kinetic-energy considerations of statistical mechanics. Thermal-noise displacement may also be classically computed from the dynamical response of the damped spring-mass system of Figure 1 driven by noise. The thermodynamic noise power N_T available per unit bandwidth to the damping element at equilibrium is simply the total thermodynamic noise power of the harmonic oscillator kT.[1,3]

$$N_T(f) = kT \quad \text{per Hertz of bandwidth} \tag{3a}$$

In terms of angular frequency, the thermal noise power per radian is given by Equation (3b).

$$N_T(\omega) = \frac{kT}{2\pi} \quad \text{per radian/s of angular bandwidth} \tag{3b}$$

The thermodynamic noise power N_T is present by virtue of the temperature of the system. This noise power is manifest in the loss elements of a system.[2] In the system of Figure 1, the damping element is the source of the thermal noise. The noise power N_T is the power that the damping element could theoretically deliver to an impedance-matched noiseless damping element. Figure 2a shows a simple "hot" damping element driving an identical "cold" noiseless damping element. The hot damping element is modeled as a noiseless damping element in parallel with a thermal force F_n. The noise power per unit bandwidth delivered to the cold element is kT as noted above. Therefore, by inspection, the rms force F_n is given by Equation (4).

$$F_n(f) = \sqrt{4kTb} \quad \text{per } \sqrt{Hz} \quad \text{rms} \tag{4a}$$

$$F_n(\omega) = \sqrt{\frac{2kTb}{\pi}} \quad \text{per } \sqrt{Radian/s} \quad \text{rms} \tag{4b}$$

The noise source may also be modeled as a displacement source in series with a damping element as shown in Figure 2b. By inspection of Figure 2b, rms displacement source X_n is given by Equation (3).

$$X_n = \frac{1}{2\pi f}\sqrt{\frac{4kT}{b}} \quad \text{per } \sqrt{Hz} \quad \text{rms} \tag{5}$$

Using the noise force model of Figure 2a, the complete noise-driven damped spring-mass harmonic oscillator is shown in Figure 3. From Newton's Second Law of Motion, force is equal to the rate of change of linear momentum.

$$F = \frac{d}{dt}(mv) = v\frac{dm}{dt} + m\frac{dv}{dt} = \dot{m}v + ma \qquad (6)$$

$$F = ma \quad \text{for } \dot{m} = 0 \qquad (7)$$

The acceleration term *a* in Equation (7) is the specific term of interest in the typical gravimeter. Therefore, Equation (7) suggests that it would be most convenient if the temporal change in mass were identically zero. This is almost a trivial observation, but nonetheless very important to formalize. There are a number of systematic effects that could result in an actual, but unexpected, change in the effective mass of the system. One of these, respiration of the proof mass, is noted above. If the proof mass is submersed in a non-vacuum environment, either gaseous or liquid, the environmental constituents will diffuse in and out of the proof mass atomic structure as a function of temperature and pressure thereby changing the effective system mass over time. Also, environmental constituents will adhere to the surface of the proof mass. This adsorption will similarly alter the effective system mass. Further, environmental constituents could actually react with the proof-mass material causing a permanent change in the proof mass. The proof mass could also be changed by evaporation where either small amounts of mass evaporate from the proof mass and condense in the environment, thereby reducing the proof mass with time, or small amounts of mass evaporate in the environment and condense on the proof mass, thereby increasing the proof mass with time. The caging mechanism could be another source of mass variation. If the caging mechanism captures the proof mass with any type of sliding motion, a small quantity of mass could be transferred either to or from the proof mass with each caging. This could have the effect of both a general trend in the change in system mass over long duration and a random change in system mass from use to use of the instrument, i.e. caging to caging. Also, if there are any contaminants in the system, such as a lubricant, this contaminant will be transferred to and from the proof mass by evaporation and condensation and with caging causing similar mass variations with time. The unexplained tares reported in the art could be related to a non-zero mass derivative. Therefore, although it may seem a trivial observation that proof-mass time derivative must be made identically zero in a precision gravimeter, actually achieving this requirement in harsh field deployment is not trivial.

For the purposes of this work, any changes in the system mass are defined as systematic errors and are not considered. The system mass is defined as constant and Equation (7) applies. Therefore, the summation of the forces in the system of Figure 1 identically equals the acceleration force experienced by the proof mass.

$$m\ddot{x} = -b\dot{x} - k_s x + F(t) \tag{8}$$

where: $\dot{x}$ and $\ddot{x}$ are the first and second time derivatives respectively

For this analysis, the initial spring displacement is defined as zero. This too is difficult to achieve in practice. The spring will creep with time changing its apparent free length. Further, the spring will be very sensitive to temperature. Achieving a constant free length in the spring is at least as challenging as maintaining a constant proof mass.

The dynamical equation of motion of the simple driven, damped, spring-mass system of Figure 1 is then given by Equation (9).

$$m\ddot{x} + b\dot{x} + k_s x = F(t) \tag{9}$$

In more canonical form,

$$\left(\frac{m}{k_s}\right)\ddot{x} + \left(\frac{b}{k_s}\right)\dot{x} + x = \left(\frac{1}{k_s}\right)F(t)$$

$$\left(\frac{1}{\omega_0^2}\right)\ddot{x} + \left(\frac{b}{k_s}\right)\dot{x} + x = \left(\frac{1}{k_s}\right)F(t) \tag{10}$$

where: $\omega_0 \equiv \sqrt{k_s/m}$ as noted above

It is convenient to define the term Q. For a system driven at its undamped natural frequency ω_0, Q is defined as 2π multiplied by the ratio of the total energy stored in the driven system to the energy dissipated over one period. It should be noted that although this definition of Q is universal, the actual expression for Q in terms of system parameters will be dependent on the physical configuration of the system.

The energy stored in a mechanical second-order system is given by both the potential energy in the spring at maximum extension (or compression) or the kinetic energy of the mass at peak velocity. The average power dissipated is simply one half the product of the peak velocity squared and the damping coefficient. The energy dissipated in one period is the product of the average power and the oscillation period, and the period of oscillation is simply $2\pi/\omega_0$.

$$E_{stored} = \frac{mv_0^2}{2} \text{ or } \frac{k_s x_0^2}{2}$$

where: v_0 = Peak Velocity

x_0 = Peak Displacement

$$P_{avg} = \frac{v_0^2 b}{2}$$

$$E_{dissipated} = \left(\frac{v_0^2 b}{2}\right)\left(\frac{2\pi}{\omega_0}\right) = \frac{\pi v_0^2 b}{\omega_0}$$

$$Q = 2\pi\left(\frac{E_{stored}}{E_{Dissipated}}\right) = \frac{m\omega_0}{b}$$

From above, $\omega_0 = \sqrt{k_s/m}$ so Q may also be expressed in terms of k_s as well,

$$Q = \frac{m\omega_0}{b} = \sqrt{\frac{m^2 k_s}{b^2 m}\left(\frac{k_s}{k_s}\right)} = \sqrt{\frac{k_s^2}{b^2}\left(\frac{m}{k_s}\right)} = \frac{k_s}{\omega_0 b} \tag{11}$$

Equation (10) may now be written in terms of *Q*.

$$\left(\frac{1}{\omega_0^2}\right)\ddot{x} + \left(\frac{1}{\omega_0 Q}\right)\dot{x} + x = \left(\frac{1}{k_s}\right)F(t)$$

$$\ddot{x} + \left(\frac{\omega_0}{Q}\right)\dot{x} + \left(\omega_0^2\right)x = \left(\frac{\omega_0^2}{k_s}\right)F(t) \tag{12}$$

$$\ddot{x} + \left(2\zeta\omega_0\right)\dot{x} + \left(\omega_0^2\right)x = \left(\frac{\omega_0^2}{k_s}\right)F(t) \tag{13}$$

where: $\zeta = \dfrac{1}{2Q}$

From control theory, Equation (13) is the generally canonical form for a second-order system. The term ω_0 is the undamped natural frequency and ζ is defined as the damping ratio of the second-order system. Note that ζ is not the same parameter as the damping coefficient *b* of the damping element. Although the expression for *Q* is system dependent, the canonical characteristic equation is common to any second-order system.

Transforming Equation (13) and solving for the magnitude of x^2:

$$\left|x(\omega)\right|^2 = F^2(\omega)\left[\frac{1/\left(\omega^2 b^2\right)}{1 + Q^2\left(\omega/\omega_0 - \omega_0/\omega\right)^2}\right]$$

Substituting Equation (4b):

$$= \left(\frac{2kT}{\pi b}\right)\left[\frac{1/\omega^2}{1+Q^2\left(\omega/\omega_0 - \omega_0/\omega\right)^2}\right] \tag{14}$$

As seen from Equation (14), the noise displacement is a function of frequency. Therefore, Equation (14) must be integrated over all positive frequency to find the total thermal-noise displacement. A more convenient form for integration is given in Equation (15).

$$\left|x(\omega)\right|^2 = \left(\frac{2kT}{\pi b}\right)\left(4\zeta^2\omega_0^2\right)\left[\frac{1}{\omega^4 + \left(4\zeta^2 - 2\right)\omega_0^2\omega^2 + \omega_0^4}\right] \tag{15}$$

Integrating Equation (15) over the range of all positive ω yields the results of Equation (16).[4]

$$\int_0^\infty \left|x(\omega)\right|^2 d\omega = x_n^2 = \frac{kT}{m\omega_0^2} = \frac{kT}{k_s} \text{ for } Q \geq 0.5 \tag{16}$$

Therefore, the same result is obtained from the dynamical solution as from the statistical mechanics solution of Equation (2), as of course must be the case. However, this is a deceiving result in terms of the performance of the spring-mass gravimeter. Even though increasing the spring constant will reduce thermal-noise displacement, the total signal-to-noise ratio will be reduced. The spring extension is the signal parameter proportional to gravity, and is the output parameter observed in the spring-mass gravimeter. When the spring constant is increased, the magnitude of spring extension due to gravity is reduced. This is a reduction in the observed signal. When the spring constant is increased, the spring extension is reduced by a factor greater than the reduction in the thermal-noise displacement which results in reduced signal-to-noise ratio even though the actual noise displacement is reduced. In order to truly optimize the system noise performance, the desired output parameter, gravity, must be found in terms of thermal noise and the system parameters.

The force F_g exerted on the spring by the proof mass due to gravity is simply *mg*. The spring force F_s due to extension (or compression) of the spring is simply $k_s x$. At equilibrium, these two forces are exactly balanced in the spring-mass gravimeter.

$$F_g = mg$$

$$F_s = k_s x$$

$$mg = k_s x \tag{17a}$$

The parameter x_n may be considered a fixed, time-invariant length which is the smallest proof-mass displacement that may be statistically observed. The change in gravity g_n required to cause an x_n change in displacement in the proof mass against the force of the spring may then be computed.

$$mg_n = k_s x_n \tag{17b}$$

$$g_n = x_n\left(\frac{k_s}{m}\right) = x_n\omega_0^2$$

$$g_n = \sqrt{\frac{kT\omega_0^2}{m}} = \sqrt{\frac{kTk_s}{m^2}} \quad \text{rms} \tag{18}$$

This term g_n is the ***gravity-equivalent noise*** of the system of Figure 3. This is the total full-bandwidth gravity-equivalent noise of the system. This result does not consider any excess noise such as cultural, systematic or 1/f noise contributions. This result as expressed in Equation (18) is much more revealing than that above in Equation (2a). Equation (18) shows that an increase in spring constant will indeed increase the gravity-equivalent noise g_n, rather than reduce noise as implied by Equation (2a) Equation (14). From Equation (18), increasing the system proof mass will reduce the gravity-equivalent noise. Since the mass term is squared, changes in the mass will have a much greater effect than changes in spring constant. Although the effect of the spring constant and the mass on the system are not orthogonal, these two parameters may be selected independently to provide a specific gravity-equivalent noise and a specific undamped natural frequency. In general, the greater the mass and the lower the undamped natural frequency the lower the gravity-equivalent noise.

The gravity-equivalent noise provided by Equation (18) is the rms noise level. Signals having peak amplitudes equal to the rms level will be obscured by the noise and will generally be unobservable. A more useful noise parameter is the tangential noise. If the noise is Gaussian, stationary, ergotic and white, the tangential noise is a factor of 2.208 greater than the rms noise.[5,6]

$$g_{\tan} = 2.208 g_n \tag{19}$$

The system parameters are known for several well-known gravimeters[7,8]: LaCoste & Romberg, GWR, Delta-g and a superconducting gradiometer described by Chan, et. al. The limiting thermodynamic gravity-equivalent noise may be computed for these instruments. These data are shown in Table 1.

Table 1. Gravimeter Noise Limits

Quantity	L&R	GWR	Delta-g	Chan
$m[g]$	10	3.0	0.050	400
$k_s[Nt/m]$	6.86x10^{-4}	6.22x10^{-3}	4.05x10^{-5}	5.70x10^{3}
$\omega_o[1/s]$	0.262	1.44	0.900	119
$T[°K]$	323	4.2	333	~4
$x_n[m_{rms}]$	2.55x10^{-9}	9.65x10^{-11}	1.07x10^{-8}	9.84x10^{-14}
$g_n[\mu Gal_{rms}]$	0.0175	0.0200	0.863	0.140
$g_{\tan}[\mu Gal]$	0.0386	0.0442	1.91	0.310

As expected from Equation (18), the L&R unit having the largest proof mass of the first three units exhibits the lowest thermal-noise displacement even though the GWR unit operates at a much lower temperature. The device described by Chan, even though it operates at a very low temperature and comprises a very large proof mass, exhibits an extremely high spring constant which results in comparatively high total gravity-equivalent noise.

The noise may be reduced by averaging the observation over a longer period. However, adding a separate filter element, whether it be mechanical, electronic or optical, will add additional noise sources and complexity to the system. Reducing the undamped natural frequency has the same effect as increasing the observation period. A lower-noise approach to reducing noise by increasing the observation period is to simply increase the proof mass. This will automatically provide additional filtering without the need to add any additional system elements.

A useful parameter to compute is the equivalent variation in elevation corresponding to the gravity-equivalent noise. The derivative of earth gravity with respect to earth radius is given by Equation (20).

$$\frac{dg}{dr} = -\frac{2g_0}{r_0} \qquad (20)$$

where: g_0 = Nominal Earth-Surface Gravity

r_0 = Nominal Earth Radius

The change in gravity with elevation at the earth surface is 0.308μGal/mm, or the change in elevation with gravity is 3.25mm/μGal. The LaCoste & Romberg instrument, using the tangential-noise limit, is capable of detecting a change in elevation of about 0.125mm based on first-principle noise limits. That distance is roughly the thickness of one sheet of the paper on which this manuscript is printed.

It is also useful to compare these results to the radial acceleration a_r at the earth surface due to the earth rotation.

$$a_r = \frac{4\pi^2 r_0}{t_d^2} \tag{21}$$

where: r_0 = Radial Radius of Rotation

t_d = Rotational Period

At the equator, the radial acceleration is 3.369Gal.

The rate of change of the radial acceleration with elevation is given simply as the derivative of the radial acceleration with respect to the radius of rotation.

$$\frac{da_r}{dr} = \left(\frac{2\pi}{t_d}\right)^2 \tag{22}$$

Therefore, at the equator, the rate of change of radial acceleration with elevation is 0.529µGal/m. This shows that the effect of the earth rotation is a significant factor. This effect will be a function of both the latitude where the system is deployed and the change in elevation that the system experiences during operation.

Another useful parameter is the equivalent change in mass corresponding to the gravity-equivalent noise. Solving Equation (17a) for *m* and substituting the values of k_s and x_n for the LaCoste & Romberg Instrument from Table 1, the change in mass that corresponds to the rms thermal-noise displacement is $1.78e^{-13}$ kg. This is a rather small mass. For a Tungsten (W) proof mass (183.85g/mole), this mass-equivalent noise is equivalent to $5.84e^{11}$ W atoms. Therefore, the mass-equivalent noise corresponds to a large number of proof-mass atoms so the system will not be at all affected by single atomic-level events. However, even though 10^{11} atoms may seem a rather large number, on the scale of human activity it is an extremely small quantity. It is very possible that much greater mass (perhaps several ηg) could be transferred to or from the proof mass with each application or release of a less-than-competent caging mechanism. Further, the absorption and adsorption of atmospheric constituents by the proof mass as well as movement of contaminants such as minute quantities of lubricants to and from the proof mass could easily result in much greater effects than those due to thermodynamic noise. If measurement precision approaching the thermal-noise limit is to be achieved, the attention to detail in the management of the spring-mass system must be meticulous.

The present precision in the state of the art in precision gravity measurement is nominally about 1μGal. The noise-equivalent gravity of the LaCoste & Romberg instrument is about two orders of magnitude below that present measurement precision. This suggests that there is very much room for improvement. It is generally relatively straight forward to effect sensing systems providing precision to within an order of magnitude of the thermodynamic noise limits, and with very careful design, sensing to within a percent or two of thermal noise limit is possible. For example, a modest quality high-fidelity sound system will exhibit a limiting system noise perhaps a factor of ten above the thermal noise, and a high-quality, low-noise satellite receiver may exhibit a limiting noise only one percent above the thermal noise limit. Therefore, thermodynamic noise is not the limiting mechanism limiting the present precision to 1μGal. If the other perturbing influences can be brought under adequate control, first-principle limitations suggest that with the LaCoste & Romberg instrument a measurement precision on the order of 0.05μGal could be possible.

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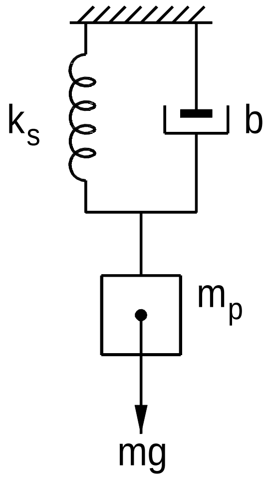

Figure 1
Simple
Harmonic Oscillator

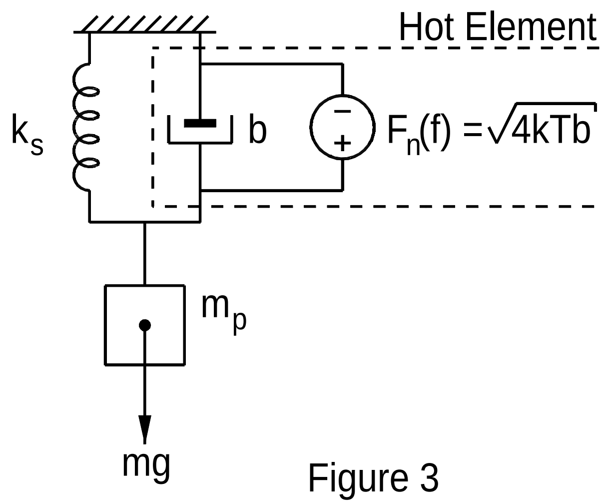

Figure 3
Noise-Driven
Harmonic Oscillator

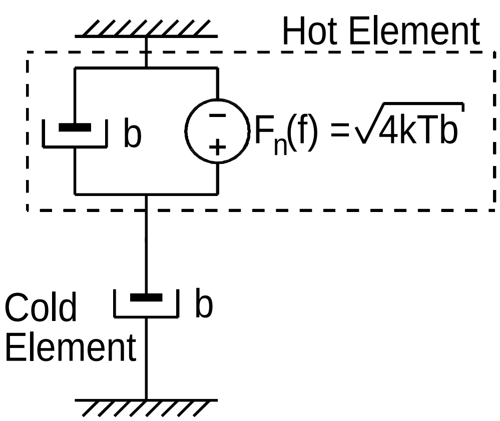

Figure 2a
Shunt Noise-Force
Model

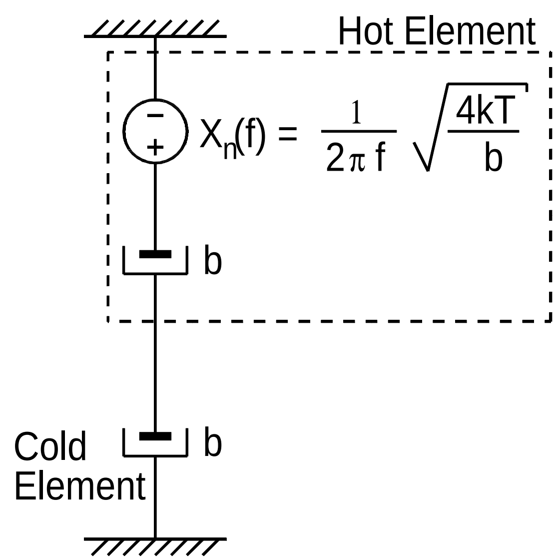

Figure 2b
Series Noise-Displacement
Model